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Identification of some source densities of the distribution type

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Abstract

In this paper we investigate the solvability of an ill-posed two-dimensional Fredholm integral equation of the first kind which allows the solutions of distribution type. The problem is first transformed into a well-posed differential–integral equation using output least-squares approach with a regularization of bounded variations. A globally convergent iterative method is proposed and some numerical results are presented. The methodology discussed may be applied for the identification of the boundary shape of the defects of a dielectric material or the interface between different materials.

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1. Introduction

Consider the Maxwell equations

$$\varepsilon(x)\boldsymbol{E}_t + \boldsymbol{J}(x,t) = \nabla \times \boldsymbol{H} \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$\mu(x)\boldsymbol{H}_t = -\nabla \times \boldsymbol{E} \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

where $\Omega \subset \mathbb{R}^3$ is occupied, for example, by a dielectric material. $\boldsymbol{E}$ and $\boldsymbol{H}$ represent the electric and magnetic fields, $\boldsymbol{J}$ is the current density. The coefficients ε and μ are the permittivity and permeability of the material. Eliminating $\boldsymbol{E}$ by differentiating (1.1) and using (1.2), we obtain

$$\mu\boldsymbol{H}_{tt} + \nabla \times \left(\frac{1}{\varepsilon}\nabla \times \boldsymbol{H}\right) = \nabla \times \left(\frac{1}{\varepsilon}\boldsymbol{J}(x,t)\right),$$

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this becomes in the steady-state case,

$$\nabla \times \left(\frac{1}{\varepsilon(x)} \nabla \times \boldsymbol{H} \right) = \nabla \times \left(\frac{1}{\varepsilon(x)} \boldsymbol{J} \right). \tag{1.3}$$

If the considered domain consists of two materials with different dielectric coefficients ε, namely $\varepsilon(\boldsymbol{x})$ is discontinuous along some interface within the domain, then equation (1.3) can be regarded as the following differential equation with a continuous coefficient but with a singular source density, i.e.,

$$\nabla \times (\nabla \times \boldsymbol{H}) = \boldsymbol{J}(x) + \boldsymbol{g}_\varepsilon(x). \tag{1.4}$$

Then the location of the interface can be determined once the singular source density $\boldsymbol{g}_\varepsilon$ is available.

Now consider a thin plate $V = (0, d_0) \times \Omega$ with d_0 being the thickness of the plate, and Ω a two-dimensional planar domain. Suppose we can make the following measurement of the eddy current at a position x with a vertical distance d from the thin plate V:

$$F(x) = \int_V \frac{x - x'}{|x - x'|^3} \times (\nabla \times \boldsymbol{H}(x'))\, \mathrm{d}x'.$$

Assume that the magnetic field is generated so that it takes the form $\boldsymbol{H} = (0, 0, H(x_1, x_2))^\top$, then the component of $F(x)$ along the vertical direction is

$$\begin{aligned} F_3(x) &= -\int_V \frac{(x_1 - x_1')H_{x_1} + (x_2 - x_2')H_{x_2}}{|x - x'|^3}\, \mathrm{d}x \\ &= \int_V \nabla_{x'} \frac{1}{|x - x'|} \cdot \nabla_{x'} H(x')\, \mathrm{d}x' \\ &= -\int_V \frac{\triangle_{x'} H(x')}{|x - x'|}\, \mathrm{d}x' + \int_{\partial V} \frac{\partial H}{\partial n}(x') \frac{1}{|x - x'|}\, \mathrm{d}x'. \end{aligned}$$

This with Eq. (1.4) and the relation

$$\nabla \times (\nabla \times \boldsymbol{H}) = (0, 0, -\triangle_{x'} H(x'))$$

gives the following inverse problem:

Find the distribution $\rho(x)$ such that

$$\int_\Omega k_d(x, x')\, \rho(x')\, \mathrm{d}x' = f(x), \quad x \in \Omega \tag{1.5}$$

for a given f, often available only in a noised form due to the measurement error. Here $k_d(x, x')$ is given by

$$k_d(x, x') = \frac{1}{\sqrt{|x - x'|^2 + d^2}}. \tag{1.6}$$

For convenience, we define an operator $K_d : L^2(\Omega) \to L^2(\Omega)$ by

$$K_d \rho(x) = \int_\Omega k_d(x, x') \rho(x')\, \mathrm{d}x'. \tag{1.7}$$

Clearly K_d is a self-adjoint operator with respect to the $L^2(\Omega)$-inner product $(\cdot,\cdot)$, i.e., $K_d^* = K_d$. Moreover, the operator K_d is bounded and satisfies

$$\|K_d\,\rho\|_{L^2(\Omega)} \leqslant k_0\,\|\rho\|_{L^2(\Omega)}. \tag{1.8}$$

with $k_0 = |\Omega|^{1/2}/d$.

The special case of the integral problem (1.5) with the zero distance, i.e., $d=0$ in (1.6), arises in finding the charge density on a thin plate using the potential measurement of the plate. In this case the kernel function is singular, the inverse problem (1.5) is not so difficult, and it was discussed numerically in [5]. In our currently interested physical situation, the distance d from the thin plate to the position where the measurement of the eddy current was made is not allowed to be zero for a few practical reasons, so the kernel $k_d(x,x')$ is a very smooth function. But the inverse problem (1.5) becomes highly ill-posed. In particular, we can allow the source density function $\rho(x)$ to be a delta-type distribution, while this is not allowed to happen in the case of $d=0$. Our main concern of this paper is to propose a stable method for the recovery of the source density of distribution type and analyse the stability and convergence of the proposed method in terms of the regularization parameter and the distance d.

One of the direct applications of this identification technique is to locate the defects or the cracks of the materials as well as the junction between different materials. Other approaches for similar identifications can be found in [1–3].

2. Regularization method and its dependence on parameters

Since $f(x)$ is often available only with some observation noise, system (1.5) may not have a solution. Even if there exist solutions, the solutions may vary unstably with respect to the changes of f. Hence we propose to use the output least-squares method with a regularization of bounded variations (BV) to solve the Fredholm integral equation (1.5):

$$\min_{\rho\in H^1(\Omega)} J(\rho) = \tfrac{1}{2}\|K_d\rho - f\|^2_{L^2(\Omega)} + \beta\,|\rho|_{\mathrm{BV}} + \frac{\mu}{2}\|\nabla\rho\|^2_{L^2(\Omega)}, \tag{2.1}$$

where $|\rho|_{\mathrm{BV}}$ is defined either by

$$|\rho|_{\mathrm{BV}} = \int_\Omega \sqrt{|\rho_{x_1}|^2 + |\rho_{x_2}|^2}\,\mathrm{d}x \tag{2.2}$$

or equivalently by

$$|\rho|_{\mathrm{BV}} = \int_\Omega |\rho_{x_1}|\,\mathrm{d}x + \int_\Omega |\rho_{x_2}|\,\mathrm{d}x. \tag{2.3}$$

In our applications, we will always fix the parameter μ as a small number, and let only β play the role of regularization in order to properly handle the noise in the data and the nonsmoothness of the function $\rho(x)$. If the solution $\rho(x)$ is smooth, one may drop off the β-term. Our major interest of the paper is to recover the nonsmooth parameter function $\rho(x)$, for example, ρ is only a delta-type distribution.

For convenience, we will use ρ^* to denote the solution of (2.1). But we may write the solution also as ρ^*_β or ρ^*_d when we want to emphasize its dependence on the regularization parameter β or on the distance d in some situations. We will frequently use the following A_d-inner product defined by

$$A_d(\sigma,\rho)=(K_d\sigma,K_d\rho)+\mu(\nabla\sigma,\nabla\rho)\quad\forall\sigma,\rho\in H^1(\Omega)$$

and its induced norm

$$\|\sigma\|_{A_d}=(A_d\sigma,\sigma)^{1/2}\quad\forall\sigma\in H^1(\Omega).$$

The L^2-norm $\|\cdot\|_{L^2(\Omega)}$ will often be written as $\|\cdot\|$.

We now prove

Lemma 2.1. *There exists a unique minimizer to the problem* (2.1).

Proof. As $J(\rho)$ is bounded below, so $j_0\equiv\min_{\rho\in H^1(\Omega)}J(\rho)$ is finite, and there exists a sequence $\{\rho_n\}\subset H^1(\Omega)$ such that $\lim_{n\to\infty}J(\rho_n)=j_0$. Therefore $\{\|\nabla\rho_n\|\}$ is bounded, which implies

$$\|\rho_n\|\leqslant C \tag{2.4}$$

for some constant C. To see (2.4), we write

$$\rho_n(x)=\hat{\rho}_n+\gamma_n(x),\qquad \hat{\rho}_n=\frac{1}{|\Omega|}\int_\Omega\rho_n(x)\,\mathrm{d}x.$$

By Friedrichs' inequality, we have

$$\|\gamma_n\|=\|\rho_n-\hat{\rho}_n\|\leqslant C_0\,\|\nabla\rho_n\|,$$

thus $\{\|\gamma_n\|\}$ is bounded, and so is $\{\|\rho_n\|\}$. Otherwise if $\{\|\rho_n\|\}$ is unbounded, then $\hat{\rho}_n=\rho_n-\gamma_n$ is also unbounded as $n\to\infty$. Using this and the boundednesss of K_d (cf. (1.8)), we have

$$\begin{aligned}J(\rho_n)&=\frac{1}{2}\int_\Omega(K_d\,\rho_n-f)^2\,\mathrm{d}x+\beta\,|\rho_n|_{\mathrm{BV}}+\frac{\mu}{2}\|\nabla\rho_n\|^2\\&\geqslant\frac{\mu}{2}\|\nabla\gamma_n\|^2+\frac{1}{4}\int_\Omega(K_d\,\hat{\rho}_n)^2\,\mathrm{d}x-\frac{1}{2}\int_\Omega(K_d\,\gamma_n-f)^2\,\mathrm{d}x\\&=\frac{\mu}{2C_0^2}\|\gamma_n\|^2_{L^2(\Omega)}+\frac{\hat{\rho}_n^2}{4}\,|\Omega|\,(K_d(1))^2-\frac{1}{2}\int_\Omega(K_d\,\gamma_n-f)^2\,\mathrm{d}x\\&\to\infty.\end{aligned}$$

This contradicts with the boundedness of $J(\rho_n)$. Therefore (2.4) is true and $\{\|\rho_n\|_{H^1(\Omega)}\}$ is bounded. So there exists a subsequence, still denoted as $\{\rho_n\}$, such that it converges weakly to a ρ^* in $H^1(\Omega)$. This with the lower semi-continuity of a seminorm, we derive

$$J(\rho^*)\leqslant\lim_{n\to\infty}\inf J(\rho_n)=\min_{\rho\in H^1(\Omega)}J(\rho),$$

so ρ^* is a minimizer.

Note that for any $\rho_1\not\equiv\rho_2$ in Ω, we have either $\nabla\rho_1\not\equiv\nabla\rho_2$ in Ω or $K_d\,\rho_1\neq K_d\,\rho_2$. Then it is easy to verify that $J(\rho)$ is strictly convex, which implies the uniqueness of the minimizers. $\square$

We now derive the optimality conditions for the minimizers in (2.1). For convenience, we let

$$F(\rho) = \tfrac{1}{2}\|K_d\,\rho - f\|^2 + \frac{\mu}{2}\|\nabla\rho\|^2, \quad j(\rho) = |\rho|_{\mathrm{BV}}.$$

Let ρ^* be a minimizer of (2.1), then

$$J(\rho^*) \leqslant J(\rho) \quad \forall\,\rho \in H^1(\Omega), \tag{2.5}$$

or equivalently

$$0 \geqslant J(\rho^*) - J(\rho) = F(\rho^*) - F(\rho) + \beta\,(j(\rho^*) - j(\rho)),$$

replacing ρ by $\rho^* + t(\rho - \rho^*)$ above for $t > 0$, and using the convexity of $j(\rho)$, we get

$$0 \geqslant t^{-1}\{F(\rho^*) - F(\rho^* + t(\rho - \rho^*)\} + \beta\,(j(\rho^*) - j(\rho)) \quad \forall \rho \in H^1(\Omega).$$

Letting $t \to 0^+$, we get

$$F'(\rho^*)(\rho^* - \rho) + \beta(j(\rho^*) - j(\rho)) \leqslant 0 \quad \forall \rho \in H^1(\Omega).$$

This gives the necessary condition of the optimality. In fact, it is also a sufficient condition, that is, we have

Theorem 2.1. *The necessary and sufficient conditions of optimality for problem* (2.1) *is*

$$F'(\rho^*)(\rho^* - \rho) + \beta\,(j(\rho^*) - j(\rho)) \leqslant 0 \quad \forall \rho \in H^1(\Omega). \tag{2.6}$$

Proof. To prove the sufficient part, let $\rho^* \in H^1(\Omega)$ be such that (2.6) holds. Then by the convexity of $F(\rho)$, we have

$$\begin{aligned} J(\rho^*) - J(\rho) &= F(\rho^*) - F(\rho) + \beta(j(\rho^*) - j(\rho)) \\ &\leqslant F'(\rho^*)(\rho^* - \rho) + \beta(j(\rho^*) - j(\rho)) \leqslant 0 \quad \forall \rho \in H^1(\Omega), \end{aligned}$$

so ρ^* is a minimizer. □

The next lemma shows the Lipschitz continuity of the minimizer function ρ^*_β with respect to the regularization parameter β:

Lemma 2.2. *The minimizer ρ^*_β of problem* (2.1) *is A_d-Lipschitz continuous with respect to the regularization parameter β. More accurately, we have*

$$\|\rho^*_\beta - \rho^*_{\beta'}\|_{A_d} \leqslant C\,|\beta - \beta'| \quad \forall\,\beta, \beta' > 0,$$

where C is a constant independent of the distance parameter d, β and β'.

Proof. For any β, $\beta' > 0$, using the optimality conditon (2.6) with ρ^* and ρ replaced by $\rho^*_{\beta'}$ and ρ^*_β, we get

$$(K_d\,\rho^*_{\beta'} - f,\ K_d\,(\rho^*_{\beta'} - \rho^*_\beta)) + \mu\,(\nabla\rho^*_{\beta'},\ \nabla(\rho^*_{\beta'} - \rho^*_\beta)) + \beta'(j(\rho^*_{\beta'}) - j(\rho^*_\beta)) \leqslant 0, \tag{2.7}$$

while replacing ρ^* and ρ by ρ^*_β and $\rho^*_{\beta'}$ in (2.6), we obtain

$$(K_d\,\rho^*_\beta - f,\ K_d\,(\rho^*_\beta - \rho^*_{\beta'})) + \mu(\nabla\rho^*_\beta, \nabla(\rho^*_\beta - \rho_{\beta'})) + \beta(j(\rho^*_\beta) - j(\rho^*_{\beta'})) \leqslant 0. \tag{2.8}$$

Summing up inequalities (2.7)–(2.8) give

$$\|K_d(\rho^*_\beta-\rho^*_{\beta'})\|^2+\mu\,\|\nabla(\rho^*_\beta-\rho^*_{\beta'})\|^2+(\beta-\beta')(j(\rho^*_\beta)-j(\rho^*_{\beta'}))\leqslant 0. \tag{2.9}$$

On the other hand, it is easy to see

$$|j(\rho^*_\beta)-j(\rho^*_{\beta'})|\leqslant\sqrt{2}\,|\Omega|^{1/2}\,\|\nabla(\rho^*_\beta-\rho^*_{\beta'})\|',$$

the desired result now follows immediately from Young's inequality and (2.9). □

For the continuity of the minimizer ρ^*_d of problem (2.1) with respect to the distance parameter d, we have

Lemma 2.3. *For any $d>d'>0$,*

$$\|K_d(\rho^*_d-\rho^*_{d'})\|+\mu^{1/2}\|\nabla(\rho^*_d-\rho^*_{d'})\|\leqslant 3\,\|f\|\frac{|d-d'|}{d'}. \tag{2.10}$$

Proof. For any $d,\ d'>0$, using the optimality condition (2.6) with ρ^* and ρ replaced by ρ^*_d and $\rho^*_{d'}$, we get

$$(K_d\rho^*_d-f,K_d(\rho^*_d-\rho^*_{d'}))+\mu(\nabla\rho^*_d,\ \nabla(\rho^*_d-\rho^*_{d'}))+\beta(j(\rho^*_d)-j(\rho^*_{d'}))\leqslant 0,$$

while replacing ρ^* and ρ by $\rho^*_{d'}$ and ρ^*_d in (2.6), we obtain

$$(K_{d'}\rho^*_{d'}-f,K_{d'}(\rho^*_{d'}-\rho^*_d))+\mu\,(\nabla\rho^*_{d'},\ \nabla(\rho^*_{d'}-\rho^*_d))+\beta\,(j(\rho^*_{d'})-j(\rho^*_d))\leqslant 0.$$

Summing up the above two inequalities gives

$$\begin{aligned}&\|K_d(\rho^*_d-\rho^*_{d'})\|^2+\mu\|\nabla(\rho^*_d-\rho^*_{d'})\|^2\\&\quad\leqslant(f,(K_d-K_{d'})(\rho^*_d-\rho^*_{d'}))-(K_d\rho^*_{d'},\ K_d(\rho^*_d-\rho^*_{d'}))+(K_{d'}\rho^*_{d'},\ K_{d'}(\rho^*_d-\rho^*_{d'}))\\&\quad=(f-K_{d'}\rho^*_{d'},\ (K_d-K_{d'})(\rho^*_d-\rho^*_{d'}))+((K_{d'}-K_d)\rho^*_{d'},\ K_d(\rho^*_d-\rho^*_{d'})).\end{aligned} \tag{2.11}$$

By definition of K_d, we can easily verify that for any $\phi\in H^1(\Omega)$,

$$\|K_d\phi-K_{d'}\phi\|\leqslant\frac{|d-d'|}{d'}\|K_d\phi\|$$

and

$$\|K_d\phi-K_{d'}\phi\|\leqslant\frac{|d-d'|}{d}\|K_{d'}\phi\|.$$

On the other hand, by taking $\rho=0$ in (2.5) we have

$$\|K_d\,\rho_d\|\leqslant\|f\|,\quad \|K_{d'}\rho_{d'}\|\leqslant\|f\|.$$

Using these estimates we derive from (2.11) that

$$\begin{aligned}&A_d(\rho^*_d-\rho^*_{d'},\ \rho^*_d-\rho^*_{d'})\\&\quad\leqslant 2\|f\|\ \|(K_d-K_{d'})(\rho^*_d-\rho^*_{d'})\|+\|(K_d-K_{d'})\rho^*_{d'}\|\|K_d(\rho^*_d-\rho^*_{d'})\|\end{aligned}$$

$$\leqslant \left(2\|f\|\frac{|d-d'|}{d'} + \frac{|d-d'|}{d}\|K_{d'}\rho_{d'}^*\|\right)\|K_d(\rho_d^* - \rho_{d'}^*)\|$$

$$\leqslant 3\|f\|\frac{|d-d'|}{d'}\{A_d(\rho_d^* - \rho_{d'}^*,\ \rho_d^* - \rho_{d'}^*)\}^{1/2},$$

from which estimate (2.3) follows immediately. □

3. An iterative solver

We now discuss an algorithm for solving the variational inequality (2.6) for the minimizer ρ^*. Recall that $\rho^* \in H^1(\Omega)$ satisfies

$$(K_d\rho^*, K_d(\rho^* - \rho)) + \mu(\nabla\rho^*,\ \nabla(\rho^* - \rho)) + \beta(j(\rho^*) - j(\rho)) \leqslant (f, K_d(\rho^* - \rho))$$

or

$$K_d^* K_d \rho^* + \mu\nabla^*\nabla\rho^* + \beta\partial j(\rho^*) = K_d^* f, \tag{3.1}$$

where K_d^* and ∇^* are the adjoints of K_d and ∇, respectively, in terms of the L^2-inner product, and ∂j denotes the subdifferential of j.

To solve this system, we are going to use an iterative method. Note that the integral operator K_d is a global operator, its discretized version is a dense matrix. To avoid solving a discretized system with a dense coefficient matrix at each iteration, we propose to use the implicit time-marching iteration. First, we solve the corresponding linear system of (3.1) for an initial value $\rho^0 \in H^1(\Omega)$:

$$K_d^* K_d \rho^0 + \mu\nabla^*\nabla\rho^0 = K_d^* f \tag{3.2}$$

(instead one may use an arbitrary initial guess $\rho^0 \in H^1(\Omega)$), then generate the sequence $\{\rho^n\}_{n=1}^{\infty} \subset H^1(\Omega)$ by solving

$$\frac{\rho^{n+1} - \rho^n}{\triangle t} + (D + \mu\nabla^*\nabla)\rho^{n+1} + \beta\partial j(\rho^{n+1}) = K_d^* f + (D - K_d^* K_d)\rho^n, \tag{3.3}$$

where D is any fixed positve-semi-definite operator such that

$$\|D\rho\|_{L^2(\Omega)} \leqslant c_0\|\rho\|_{L^2(\Omega)} \quad \forall\rho \in L^2(\Omega) \tag{3.4}$$

for some constant c_0.

In the following, we are going to show the global convergence of the algorithm (3.3). To do so, we first give a stability estimate for the sequence $\{\rho^n\}$.

Lemma 3.1. *For the sequence generated by* (3.3) *with an arbitrary guess* $\rho^0 \in H^1(\Omega)$, *we have for any* $M > 1$,

$$\left(\frac{1}{\Delta t} - \frac{k_0^2}{2}\right)\sum_{n=1}^{M-1}\|\rho^{n+1} - \rho^n\|^2 + ((K_d^* K_d + \mu\nabla^*\nabla)\rho^M, \rho^M)$$

$$+ \sum_{n=0}^{M-1}((2D + \mu\nabla^*\nabla)(\rho^{n+1} - \rho^n), \rho^{n+1} - \rho^n) + \beta j(\rho^M) \leqslant C(\rho^0), \tag{3.5}$$

where $C(\rho^0)$ *is a constant depending on* ρ^0.

Proof. Let $L=D+\mu\nabla^*\nabla$, and $\|\cdot\|_L$ for its induced norm $(L\cdot,\cdot)^{1/2}$. Multiplying (3.3) by $(\rho^{n+1}-\rho^n)$, we get

$$\frac{1}{\Delta t}\|\rho^{n+1}-\rho^n\|^2+(L\rho^{n+1},\rho^{n+1}-\rho^n)+\beta(j(\rho^{n+1})-j(\rho^n))$$
$$\leqslant (K_d^*f,\rho^{n+1}-\rho^n)+((D-K_d^*K_d)\rho^n,\rho^{n+1}-\rho^n)$$

or

$$\frac{1}{\Delta t}\|\rho^{n+1}-\rho^n\|^2+\tfrac{1}{2}\|\rho^{n+1}\|_L^2+\tfrac{1}{2}\|\rho^{n+1}-\rho^n\|_L^2+\beta j(\rho^{n+1})$$
$$\leqslant \tfrac{1}{2}\|\rho^n\|_L^2+\beta j(\rho^n)+(K_d^*f,\rho^{n+1}-\rho^n)$$
$$+\tfrac{1}{2}((K_d^*K_d-D)\rho^n,\rho^n)-\tfrac{1}{2}((K_d^*K_d-D)\rho^{n+1},\rho^{n+1})$$
$$+\tfrac{1}{2}((K_d^*K_d-D)(\rho^n-\rho^{n+1}),\rho^n-\rho^{n+1}),$$

summing over $n=0,1,\ldots,M-1$, we obtain

$$\frac{1}{\Delta t}\sum_{n=0}^{M-1}\|\rho^{n+1}-\rho^n\|^2+\tfrac{1}{2}((L+K_d^*K_d-D)\rho^M,\rho^M)$$
$$+\tfrac{1}{2}\sum_{n=0}^{M-1}((L+D-K_d^*K_d)(\rho^{n+1}-\rho^n),\rho^{n+1}-\rho^n)+\beta j(\rho^M)$$
$$\leqslant \tfrac{1}{2}(L\rho^0,\rho^0)+\beta j(\rho^0)+(K_d^*f,\rho^M)-(K_d^*f,\rho^0)+\tfrac{1}{2}((K_d^*K_d-D)\rho^0,\rho^0).$$

Using (1.8), we further derive

$$\left(\frac{1}{\Delta t}-\frac{k_0^2}{2}\right)\sum_{n=0}^{M-1}\|\rho^{n+1}-\rho^n\|^2+\tfrac{1}{2}((K_d^*K_d+\mu\nabla^*\nabla)\rho^M,\rho^M)$$
$$+\frac{1}{2}\sum_{n=0}^{M-1}((2D+\mu\nabla^*\nabla)(\rho^{n+1}-\rho^n),\rho^{n+1}-\rho^n)+\beta j(\rho^M)$$
$$\leqslant \tfrac{1}{2}(L\rho^0,\rho^0)+\beta j(\rho^0)-(K_d^*f,\rho^0)+\frac{k_0^2}{2}\|\rho^0\|^2$$
$$-\tfrac{1}{2}(D\rho^0,\rho^0)+(K_d^*f,\rho^M). \tag{3.6}$$

Using the same argument as in proving (2.4), we have

$$\|\rho^M\|\leqslant C((K_d^*K_d+\mu\nabla^*\nabla)\rho^M,\rho^M)^{1/2}. \tag{3.7}$$

To see this, we write

$$\rho(x)=\hat\rho+\gamma(x)\quad\forall x\in\Omega,$$

where $\hat\rho$ is the average of the function ρ in Ω, then we get from (1.8) that

$$\|K_d\rho\|^2\geqslant\tfrac{1}{2}\|K_d\hat\rho\|^2-\|K_d\gamma\|^2\geqslant\frac{\hat\rho^2}{2}|\Omega|(K_d(1))^2-\frac{|\Omega|}{d^2}\|\gamma\|^2,$$

which implies

$$\hat{\rho}^2 \leqslant C(\|\gamma\|^2 + \|K_d\rho\|^2).$$

Then (3.7) follows from the triangle and Poincaré inequalities. (3.7) with Young's inequality gives

$$\begin{aligned}(K_d^* f, \rho^M) &\leqslant C((K_d^* K_d + \mu\nabla^*\nabla)\rho^M, \rho^M)^{1/2}\|K_d^* f\| \\ &\leqslant \tfrac{1}{4}((K_d^* K_d + \mu\nabla^*\nabla)\rho^M, \rho^M) + C\|K_d^* f\|^2,\end{aligned}$$

with which (3.6) leads to (3.5). □

Applying the stability estimate in Lemma 3.1 we can prove the following global convergence of algorithm (3.3):

Theorem 3.1. *Let $\{\rho^n\}_{n=1}^{\infty}$ be the iterative sequence generated by algorithm* (3.3) *with an arbitrary initial guess $\rho^0 \in H^1(\Omega)$, not necessarily given by* (3.2), *then it converges strongly in $H^1(\Omega)$ to the solution ρ^* of* (3.1).

Proof. We know from (3.5) that $\{\rho^n\}$ is bounded in $H^1(\Omega)$, therefore there exists a subsequence, still denoted as $\{\rho^n\}$, such that

$$\rho^n \to \bar{\rho} \text{ weakly in } H^1(\Omega), \quad \rho^n \to \bar{\rho} \text{ strongly in } L^2(\Omega). \tag{3.8}$$

Next, we show that $\bar{\rho}$ is a solution of (3.1). To see this, multiplying (3.3) by $(\rho^{n+1} - \rho)$, with any $\rho \in H^1(\Omega)$, we obtain

$$\begin{aligned}&\frac{1}{\Delta t}(\rho^{n+1} - \rho^n, \rho^{n+1} - \rho) + ((D + \mu\nabla^*\nabla)\rho^{n+1}, \rho^{n+1} - \rho) \\ &\quad + (K_d^* K_d \rho^n, \rho^{n+1} - \rho) + \beta j(\rho^{n+1}) - \beta j(\rho) \\ &\leqslant (K_d^* f, \rho^{n+1} - \rho) + (D\rho^n, \rho^{n+1} - \rho).\end{aligned} \tag{3.9}$$

Again from (3.5), we know that

$$\sum_{n=1}^{\infty} \|\rho^{n+1} - \rho^n\|^2 \leqslant C,$$

therefore $\|\rho^{n+1} - \rho^n\| \to 0$ as $n \to \infty$. Using this, (3.8)–(3.9) and the lower semi-continuity of a seminorm, we derive by taking the $\liminf_{n\to\infty}$ in (3.9) that

$$\begin{aligned}&(K_d\bar{\rho}, K_d(\bar{\rho} - \rho)) + \mu(\nabla\bar{\rho}, \nabla(\bar{\rho} - \rho)) + (D\bar{\rho}, \bar{\rho} - \rho) + \beta(j(\bar{\rho}) - j(\rho)) \\ &\quad \leqslant \liminf_{n\to\infty}\{(K_d^* f, \rho^{n+1} - \rho) + (D\rho^n, \rho^{n+1} - \rho)\} \\ &\quad = (f, K_d(\bar{\rho} - \rho)) + (D\bar{\rho}, \bar{\rho} - \rho) \quad \forall \rho \in H^1(\Omega).\end{aligned}$$

so $\bar{\rho}$ is the unique solution ρ^* of (2.6) or (3.1).

Using the above result, we can easily show that each subsequence of $\{\rho^n\}_{n=1}^{\infty}$ has a subsequence which converges to ρ^*. So the whole sequence $\{\rho^n\}_{n=1}^{\infty}$ converges to the same limit ρ^*. We next prove that ρ^n converges to ρ^* strongly in $H^1(\Omega)$. To do so, taking $\rho=\rho^*$ in (3.9) and summing the resulting inequality with (3.1) with ρ replaced by ρ^{n+1}, we obtain

$$\frac{1}{\Delta t}(\rho^{n+1}-\rho^n,\rho^{n+1}-\rho^*)+((D+\mu\nabla^*\nabla)(\rho^{n+1}-\rho^*),\rho^{n+1}-\rho^*)$$
$$\leqslant((D-K_d^*K_d)(\rho^n-\rho^*),\rho^{n+1}-\rho^*),$$

or equivalently

$$\frac{1}{\Delta t}\|\rho^{n+1}-\rho^*\|^2+\mu\|\nabla(\rho^{n+1}-\rho^*)\|^2\leqslant\left(c_0+k_0+\frac{1}{\Delta t}\right)\|\rho^n-\rho^*\|\|\rho^{n+1}-\rho^*\|.$$

This with the previously proved L^2-convergence of $\{\rho^n\}$ implies the desired strong convergence in $H^1(\Omega)$. □

4. Choice of regularization parameters

In this section we study the possibility of the use of Morozov's principle to choose a reasonable regularization parameter β in (2.1). Assume that the available observation data is f^δ, instead of the exact data f, and the noise level is of order δ, namely

$$\|f^\delta-f\|_{L^2(\Omega)}\leqslant\delta. \tag{4.1}$$

Let $\hat\rho\in H^1(\Omega)$ be a solution of $K_d\hat\rho=f$, and ρ_β^δ be the unique solution of the following minimization problem:

$$\min_{\rho\in H^1(\Omega)} J(\rho)=\tfrac12\|K_d\rho-f^\delta\|^2+\beta j(\rho)+\frac{\mu}{2}\|\nabla\rho\|^2_{L^2(\Omega)}. \tag{4.2}$$

Recall that for our interest in this paper, μ is fixed and much smaller than β.

The damped Morozov principle proposes that the regularization parameter β should be chosen such that the error due to the regularization is equal to the error due to the observation data. That is, β is chosen according to

$$\|K_d\rho_\beta^\delta-f^\delta\|^2+\mu\|\nabla\rho_\beta^\delta\|^2=\delta^2. \tag{4.3}$$

For such a choice of the regularization parameter β, we have the following error estimate between ρ_β^δ and $\hat\rho$:

Theorem 4.1. *If β is chosen according to* (4.3), *then we have*

$$\|K_d(\rho_\beta^\delta-\hat\rho)\|^2+\mu\|\nabla(\rho_\beta^\delta-\hat\rho)\|^2\leqslant2(\delta^2+\mu\|\nabla\hat\rho\|^2). \tag{4.4}$$

Proof. We see from (4.3) that

$$\|K_d\rho_\beta^\delta-f^\delta\|^2+\mu\|\nabla\rho_\beta^\delta\|^2-\{\|K_d\rho_0^\delta-f^\delta\|^2+\mu\|\nabla\rho_0^\delta\|^2\}\leqslant\delta^2. \tag{4.5}$$

Now we claim that the left-hand side of (4.5) is equal to $A_d(\rho_\beta^\delta-\rho_0^\delta,\rho_\beta^\delta-\rho_0^\delta)$. In fact,

$$\mathrm{LHS}=\|K_d(\rho_\beta^\delta-\rho_0^\delta)\|^2+\mu\|\nabla(\rho_\beta^\delta-\rho_0^\delta)\|^2$$
$$+2\{(K_d\rho_0^\delta-f^\delta,K_d(\rho_\beta^\delta-\rho_0^\delta))+\mu(\nabla\rho_0^\delta,\nabla(\rho_\beta^\delta-\rho_0^\delta))\}. \tag{4.6}$$

Note that ρ_0^δ is a minimizer of problem (4.2) with $\beta=0$, that is, ρ_0^δ is a solution of the following variational problem:

$$(K_d\rho_0^\delta, K_d(\rho_0^\delta-\sigma))+\mu(\nabla\rho_0^\delta,\nabla(\rho_0^\delta-\sigma))=(f^\delta,K_d(\rho_0^\delta-\sigma))\quad\forall\sigma\in H^1(\Omega), \tag{4.7}$$

now the claim follows from (4.6) and (4.7).

We next estimate $A_d(\rho_0^\delta-\hat\rho,\rho_0^\delta-\hat\rho)$. To do so, we add up (4.7) with σ replaced by $\hat\rho$ and the following equation:

$$(K_d\hat\rho-f,K_d(\hat\rho-\rho_0^\delta))=0$$

to obtain

$$\|K_d(\hat\rho-\rho_0^\delta)\|^2+\frac{\mu}{2}\|\nabla(\hat\rho-\rho_0^\delta)\|^2+\frac{\mu}{2}\|\nabla\rho_0^\delta\|^2-\frac{\mu}{2}\|\nabla\hat\rho\|^2=(f^\delta-f,K_d(\rho_0^\delta-\hat\rho)),$$

then the Cauchy–Schwarz inequality gives

$$A_d(\hat\rho-\rho_0^\delta,\hat\rho-\rho_0^\delta)\leqslant\mu\|\nabla\hat\rho\|^2+\|f^\delta-f\|^2\leqslant\delta^2+\mu\|\nabla\hat\rho\|^2. \tag{4.8}$$

Now result (4.4) follows immediately from the trianlge inequality and (4.6) and (4.8). $\square$

5. Numerical experiments

In this section we show some numerical experiments using the iterative algorithm (3.3) proposed in Section 3 for the identification of some source densities of distribution type. Let us first discuss the discretization of Eq. (3.3) for finding the $(n+1)$th iterate ρ^{n+1}. It is easy to verify that the solution ρ^{n+1} solves equivalently the following problem:

$$\min_{\rho\in H^1(\Omega)}\tilde J(\rho)=\tfrac{1}{2\Delta t}\|\rho\|^2+\tfrac12(D\rho,\rho)+\frac{\mu}{2}\|\nabla\rho\|^2+\beta j(\rho)-(\tilde f,\rho),$$

or the following nonlinear elliptic problem:

$$\frac{1}{\Delta t}\rho+D\rho-\mu\Delta\rho-\beta\nabla\cdot\left(\frac{\nabla\rho}{\sqrt{|\nabla\rho|^2+\varepsilon}}\right)=\tilde f \tag{5.9}$$

when we use (2.2) or

$$\frac{1}{\Delta t}\rho+D\rho-\mu\Delta\rho-\beta\sum_{i=1}^{2}\left(\frac{\rho_{x_i}}{\sqrt{|\rho_{x_i}|^2+\varepsilon}}\right)_{x_i}=\tilde f \tag{5.10}$$

when we use (2.3). Here ε is a small positive parameter introduced to smooth the nondifferentiable functional $j(\beta)$, and $\tilde f$ is given by

$$\tilde f=\frac{1}{\Delta t}\rho^n+K_d^*f+(D-K_d^*K_d)\rho^n. \tag{5.11}$$

We will complement Eq. (5.9) or (5.10) with the homogeneous Neumann boundary condition $\partial\rho/\partial\boldsymbol{n}=0$ on $\partial\Omega$.

Without loss of generality, we take Ω to be the unit square $\Omega=(0,1)\times(0,1)$ and the operator $D=0$. We divide the domain Ω into N^2 subsquares Ω_{ij} with each side having equal length h. Let

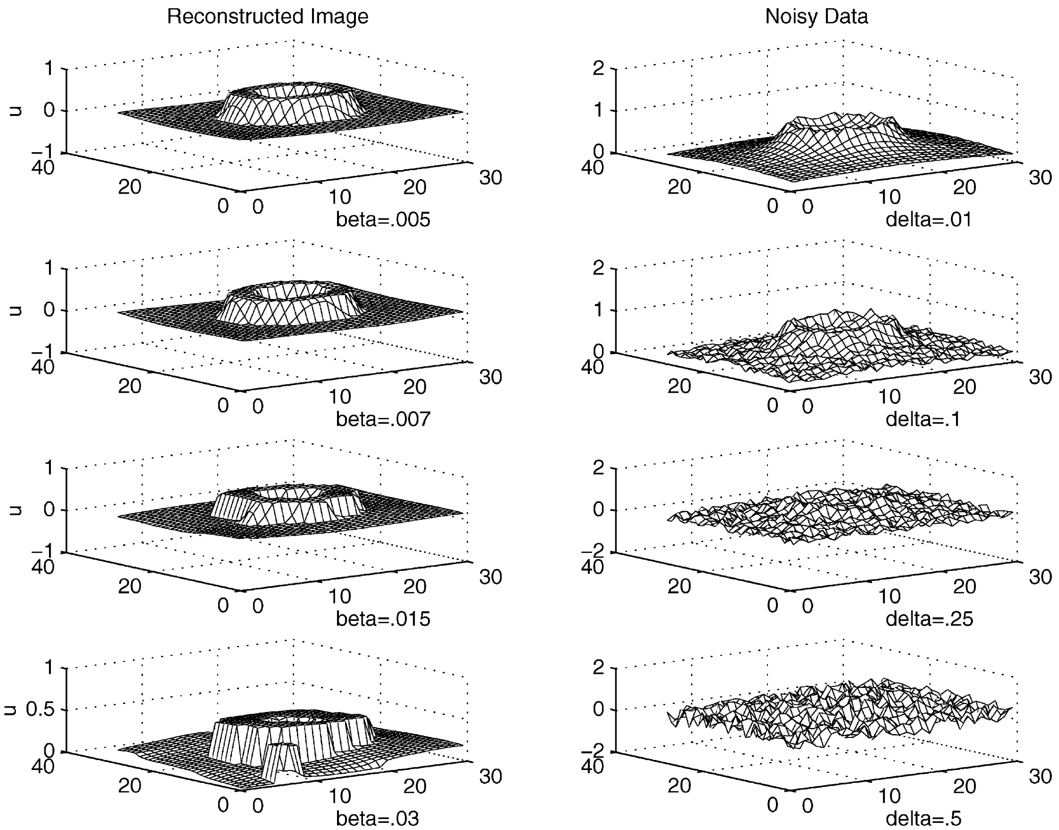

Fig. 1. Mesh 30 × 30, $\Delta t = 0.25$, $\mu = 5.0 \cdot 10^{-5}$, $\varepsilon = 10^{-10}$, $d = 0.01$.

$x_{ij} = (x_1^i, x_2^j)$ be the centroid of each Ω_{ij}. We will approximate the product of K_d with a given function ρ as follows:

$$K_d\rho(x_{lm}) = \sum_{ij}\int_{\Omega_{ij}} \frac{\rho(x')}{\sqrt{|x-x'|^2+d^2}}\,\mathrm{d}x' \approx h^2 \sum_{ij} \rho_{ij} k_{ij}(x_{lm}), \quad \forall x_{lm}.$$

We use the midpoint rule for our calculations. Thus, $k_{ij}(x_{lm})$ is the average values of $k(x_{lm}, x')$ at four vertices of Ω_{ij} and ρ_{ij} approximates ρ at x_{ij}.

We use the central difference approximation of $\Delta\rho$ at x_{ij}:

$$\frac{1}{h^2}(4\rho_{i,j} - \rho_{i+1,j} - \rho_{i-1,j} - \rho_{i,j+1} - \rho_{i,j-1})$$

and the following second-order approximation for $j(\rho)$ at $x_{i,j}$:

$$\sqrt{\frac{1}{2}\left\{\left|\frac{\rho_{i+1,j}-\rho_{i,j}}{h}\right|^2 + \left|\frac{\rho_{i,j}-\rho_{i-1,j}}{h}\right|^2\right\} + \frac{1}{2}\left\{\left|\frac{\rho_{i,j+1}-\rho_{i,j}}{h}\right|^2 + \left|\frac{\rho_{i,j}-\rho_{i,j-1}}{h}\right|^2\right\}}.$$

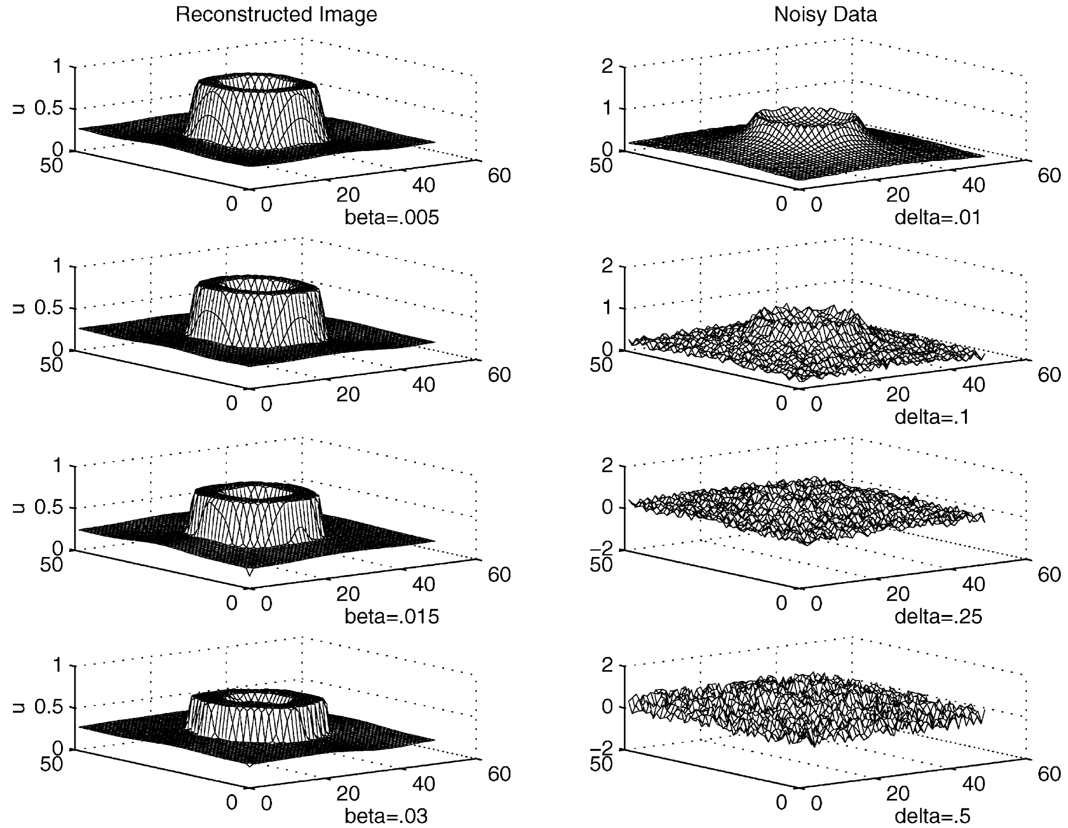

Fig. 2. Mesh 60×60, $\Delta t = 1.0$, $\mu = 5.0 \cdot 10^{-5}$, $\varepsilon = 10^{-10}$, $d = 0.01$.

Hence the discretized problem of (5.9) is given by

$$\frac{1}{\Delta t}\rho^{n+1} + \mu H \rho^{n+1} + \frac{\beta}{2}((D_1^+)^t \Lambda D_1^+ + (D_1^-)^t \Lambda D_1^- + (D_2^+)^t \Lambda D_2^+ + (D_2^-)^t \Lambda D_2^-)\rho^{n+1}$$

$$= \frac{1}{\Delta t}\rho^n + K_d f - K_d^* K_d \rho^n, \tag{5.12}$$

where Λ is the diagonal matrix with diagonal

$$\lambda_{i+(j-1)n} = \{\gamma_{ij}(\rho^{n+1}) + \varepsilon\}^{-1/2}$$

with

$$\gamma_{ij}(\rho^{n+1}) = \frac{1}{2h^2}(|\rho_{i,j}^{n+1} - \rho_{i-1,j}^{n+1}|^2 + |\rho_{i+1,j}^{n+1} - \rho_{i,j}^{n+1}|^2) + \frac{1}{2h^2}(|\rho_{i,j}^{n+1} - \rho_{i,j-1}^{n+1}|^2 + |\rho_{i,j+1}^{n+1} - \rho_{i,j}^{n+1}|^2)$$

and H denotes the central difference matrix, and

$$D_1^+ = D^+ \otimes I, \quad D_1^- = D^- \otimes I, \quad D_2^+ = I \otimes D^+, \quad D_2^- = I \otimes D^-.$$

If $n = 1/h$, then D^+ and D^- are the $n \times n$ forward and backward difference matrices respectively.

In our implementations, we use the fixed-point iterative method to solve the nonlinear algebraic system of Eqs. (5.12), namely the diagonal matrix Λ is evaluated at the previous iterate ρ^n, but we let this process iterate only once to three times. Hence, the resulting system becomes a linear system with a sparse (block tri-diagonal) symmetric positive-definite matrix and can be efficiently solved by the Cholesky decomposition method. In our numerical calculations we iterate the fixed-point iteration once. We note that we can prove the global convergence of the resulting algorithm using the similar arguments as in Section 3, e.g., see, [4].

In the example shown below, we choose the exact solution of the integral equation (1.5) to be the delta function $\delta_\Gamma(x)$ with Γ being the circle of radius 0.25 centered at $(0,5,0.5)$. Then the exact observation data f is calculated through Eq. (1.5) using the exact solution. In our implementation, we add a random noise to the observation data f in the following way:

$$f^\delta(x) = f(x) + \delta\,\mathrm{rand}(x),$$

where $\mathrm{rand}(x)$ is a uniformly distributed random function in $[-1,1]$, and δ is the noise level (Figs. 1–2).

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